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Optimal Risk-Exposure Management with Costly Refinancing Opportunities

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OPTIMAL RISK–EXPOSURE MANAGEMENT WITH COSTLY REFINANCING OPPORTUNITIES

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Abstract. In this paper the decisions of a firm's manager, in terms of exposure to a profitable but risky technology, distribution of dividends and (costly) re–injection of cash to ward off bankruptcy are studied. The analysis of the manager's optimal choices is done via a value function whose state variable is the firm's current level of reserves. Contingent on whether proportional or fixed costs of reinvestment are considered, singular stochastic control or stochastic impulse control techniques are used.

1. Introduction

This aim of this work is to study the risk–exposure decisions of a firm's manager, together with his choice of distribution of dividends, in a framework where the firm's shareholders have the choice of refinancing the firm should its cash reserves become non–positive. The manager chooses, continuously, an exposure strategy $a \in [0, 1]$ that multiplies an arithmetic Brownian motion and determines how risky/profitable the firm is at each date. Whenever the firm's cash reserves become negative it must be either liquidated (at a zero–recovery rate) or it must be refinanced. Due to the presence of refinancing costs, cash re–injection is only exercised to ward off bankruptcy. This process may be interpreted as well as costly issuance of equity. Both the cases of proportional and fixed costs are considered. The former requires tools from Singular Stochastic Control, whereas for the latter Impulse Control techniques are employed. At each point in time, the manager maximizes the value of the firm, which is a function of its current level of cash reserves, and corresponds to the discounted dividend stream net of re–investment costs (there are no informational asymmetries issues considered in our model). In contrast with the classical Merton problem, here investors are assumed to be risk neutral. This implies that the set of feasible consumption–contributions strategies cannot be assumed to consist of absolutely continuous processes. To the best of our knowledge, Radner & Shepp [13] and Jeanblanc–Piqué & Schiryaev [8] were the first ones to work in such a setting. In their case, the risky project has a fixed size and no re–investment is possible. The manager then faces the compromise of distributing too many dividends, which is inefficient due to bankruptcy risk, or postponing distribution too long. The latter option prolongs the project's lifespan, but it is inefficient because of discounting. The authors show the optimal strategy is the following: refrain from consumption as long as the cash reserves remain under a certain threshold x^*; whenever the current reserves level x is greater than x^*, consume $x - x^*$ immediately. Since

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the level of cash reserves follows a diffusion process, this results in a localized optimal strategy. The solution of the manager's problem may be, therefore, identified with a Skohorod problem, where the cash–reserves process is reflected at level x^*. The mathematical methodologies for this kind of problems have been studied, for example, in [5] and [3]. The two main departures from [13] and [8] found in the literature are:

- **Introducing the possibility of varying the size of the risky project.** Højgaard and Taksar study in [7] a model of an insurance firm whose manager controls the firm's risk exposure via proportional reinsurance. This is translated into a stochastic control problem where reinsurance is represented by a factor $1-\alpha \in [0,1]$. The second control variable, the cumulative distribution of dividends, keeps the firm's cash reserves below the optimal dividend–distribution barrier x^*. The possibility of continuously tweaking the reinsurance level results in two additional (relative to [8]) boundary points $0 < x^{***} < x^{**} < x^*$. Below x^{***} there is partial reinsurance and on (x^{***}, x^{**}) there is none. In the partial–reinsurance region, the optimal choice α^* results in a reserves level that follows a geometric Brownian motion: the proportion of reinsurance tends to one as the reserves approach zero, thus preventing the firm from going bankrupt. The model could also be interpreted as a Merton–style problem of optimal portfolio design, where α represents the proportion of wealth invested in the risky asset, and where the savings account pays zero interest. Rochet–Villeneuve [15] address a similar optimal portfolio problem, but from the point of view of a corporation. They assume the firm's debt level is fixed and its reserves may not become negative. As a consequence the amount of cash that may be invested in the risky asset is bounded above by the firm's current liabilities. Notice the departure from a proportional scenario in [7] to one in nominal terms. Here debt and reserves accrue interest at the same (fixed) rate. This feature, together with the non–proportional setting, results in a value function whose corresponding HJB variational inequality cannot be solved in closed form. This issue notwithstanding, the authors are able to show that the manager's optimal strategy is quite similar to that in [7]: invest a multiple of the firm's equity into the risky asset, keep the rest as cash reserves, and distribute dividends when the value of the firm exceeds some threshold. The resulting value function is rational with exponent smaller than one for low levels of equity, which the authors interpret as corporate risk aversion stemming from the risk of bankruptcy. Interestingly, even though the HJB variational inequality in [15] is more complicated than that in [7] and cannot be solved in closed form, the structures of the solutions to both problems are quite similar.
- **Introducing reinvestment possibilities.** Natural extensions to [8] were considered by Lokka and Zervos in [12] and by Déscamps, Mariotti, Rochet and Villeneuve in [4]. These authors allow for additional equity issuance under proportional or fixed transaction costs, respectively. In both cases closed form solutions for the firm's optimal balance sheet are given, and conditions for the viability of costly equity issuance are found. Furthermore, in [4] the impact of financial frictions on firm governance in terms of the model's predictions is studied, as well as how these governance issues affect the volatility of stock returns. The fixed–transaction–costs setting is further extended by Akyildirim, Güney, Rochet and Soner in [1], where interest rates and issuance costs are governed by an exogenous Markov chain. This provides insight, in a stylized model, on how the business cycle affects managerial decisions and, mapping back to [4], stock price

volatility. Jiang and Pistorius, in [9], analyze a model quite similar to [4], where instead of (stochastic) reinvestment opportunities, the drift in the firm's risky technology also follows a Markov chain.

To the best of our knowledge, He–Liang [6] were the first ones to study the mixed case of a firm whose manager may vary the exposure to the risky technology and may also decide to issue additional equity. Their paper, situated within the insurance literature, is the most similar to ours. They, however, take the boundaries x^{***}, x^{**} and x^{*} from [7] as given and then analyze under which conditions it is optimal to re–issue equity in order to thwart off bankruptcy. Only proportional issuance costs are considered. We also work in a model where the manager may proportionally adjust the firm's exposure to a risky production technology; however, in contrast with [6], we solve the free boundary problems of finding the partial– and full–exposure and dividend boundaries as a function of the re–investment costs, which may be proportional or fixed. Indeed, we show that when the partial exposure boundary is optimally adjusted to a proportional transaction cost, it is never the case that re–investment/re–issuance is inefficient. We have chosen the proportional setting without a (profitable) saving technology because it allows for closed–form solutions to the value functions and, more interestingly, for (almost) fully analytical expressions for the free boundaries. As a consequence, we provide conditions on the re–investment costs under which partial exposure disappears, and we are also able to study the convergence properties of the value functions to the first–best case (when costs decrease) and to the solution in [7] (when reinvestment becomes prohibitively costly). Our model extends [7] by considering re–investment possibilities and [4] and [12] by allowing a variable exposure to the risky investment opportunity.

The remainder of the paper is organized as follows: We describe in Section 2 the Basic Model (Højgaard's and Taksar's) , without re–investment options, which we use as an initial building block. The case of proportional costs of re–investment is studied in Section 3, whereas fixed–costs are introduces in Section 4. For the sake of completeness, brief description of the solution to the Basic Model is provided in the Appendix, where all our mathematical proofs can also be found.

2. The Basic Model

We work in a continuous–time, infinite horizon setting and consider a firm whose manager has the possibility to continuously modify the firm's exposure to a profitable but risky investment technology. We rule out any information asymmetries between management and equity holders; therefore, the manager's sole objective is to maximize the firm's value. The latter is defined as the expected, discounted dividend stream that the (risk–neutral) shareholders receive over the firm's lifespan.

In order to describe the dynamics of the cashflows generated by the exposure to the risky technology, as well as the dividend payments, let us introduce the filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}\big)$. The process $W(t)$ is a $\mathbb{P}$–Brownian motion that generates the filtration $\{\mathcal{F}_t\}$. Let μ, s_0 and σ be greater than zero, then the cashflows generated by full–risk exposure are given by the SDE

$$dS(t) = \mu dt + \sigma dW(t), \quad S(0) = s_0.$$

The manager's decisions regarding risk exposure are represented by the predictable process $\{a(t) \in [0, 1]\}$, which we shall call an *exposure strategy.* A *cumulative dividends process* is any non–decreasing, adapted and càglàd process $\{L(t)\}$. Even in the presence of refinancing possibilities (which we introduce in Sections 3 and 4), we restrict ourselves to non–negative dividends and model contributions separately. As a consequence, any required additional funding must be either external or provided by the shareholders. The dynamics of the firm's *cash reserves* are described by the SDE

$$dR^{a,L}(t) = a(t)(\mu dt + \sigma dW(t)) - dL(t), \quad R^{a,L}(0) = x.$$

Since, a priori, s_0 needs not match x, an exceptional dividend being distributed at zero is not excluded. Notice that we have assumed, for simplicity, that no interest is earned on cash reserves[1]. The firm is shut down when its cash reserves run out, and we consider only the case of a zero recovery rate. More specifically, the bankruptcy (stopping) time τ associated to a pair (a, L) is defined as

$$\tau := \inf\{t > 0 \mid R^{a,L} < 0\}.$$

We have used a strict inequality in the definition of τ to allow, in the upcoming sections, for reinvestment in the firm after its cash reserves hit zero. Let us denote by ρ the time–preference rate of the investors. The expected, discounted dividend stream is given by the quantity

$$\mathbb{E}_x\Big[\int_0^\tau e^{-\rho t} dL(t)\Big]. \tag{1}$$

Here $\mathbb{E}_x[\cdot]$ is the expectation operator conditional on $R^{a,L}(0) = x$. Although à priori the equity holders are risk–neutral, the risk of bankruptcy implies that they will exhibit CRRA–preferences for low levels of cash reserves. The stochastic control problem of choosing (a, L) as to maximize Expression (1) has been studied in [7]. We provide a brief summary in Appendix B for completeness, and denote the corresponding value function by $U(\cdot)$. In the sequel we refer to this setting as the *basic model*.

Below we introduce the requirement/possibility of refinancing (which may be viewed as costly equity issuance) in the form of a cumulative *contributions process* $\{G(t)\}$ (thus, we are moving away from self–financing strategies). The latter is also assumed to be non–decreasing and càglàd, and it wards off bankruptcy. The manager's task is to devise a strategy $\big\{\pi(t) = \big(a(t), L(t), G(t)\big)\big\}$ in order to maximize the expected, discounted dividend stream net of contributions. The way in which one models the cost of additional financing affects, unsurprisingly, the manager's decisions. In Section 3 we study the proportional–costs setting, whereas in Section 4 we look at the case where there is a fixed cost associated to refinancing.

3. Proportional Costs of Refinancing

In this section we study the case where there are proportional costs associated to raising additional cash for the firm. We consider a *cost factor* $\beta \geq 1$, i.e. in order to inject x units of cash into the firm, βx must be raised. Initially we assume the firm must be rescued whenever cash reserves become negative, and that at no other time are additional funds injected. We then show in Propositions 3.5 and 3.7 that this behavior is optimal.

[1]Otherwise the differential equation arising in the analysis of the value function cannot be solved explicitly. This is the case in [15], but in our setting a non–stochastic short rate results in increased mathematical complexity without adding much value in terms of the economic theory. Introducing a stochastic short rate, however, results in PDEs whose solutions must be estimated numerically. This falls outside the scope of this paper, but we refer the interested reader to [2], where some analysis in this direction is made.

Given a strategy $\pi = (a, L, G)$ and an initial cash level x, the corresponding cash–reserves process evolves according to the SDE

$$dR^{\pi}(t) = dR^{a,L}(t) + dG(t), \quad R^{\pi}(0) = x,$$

and the corresponding expected, discounted value of the firm (in terms of dividends net of contributions) is

$$V^{\pi}(x) = \mathbb{E}_x\Big[\int_0^{\infty} e^{-\rho t}(dL(t) - \beta dG(t))\Big].$$

We note that in the expression above, the integral's upper limit has become ∞ as a result of the mandatory refinancing strategy that we have imposed for the time being. We define the set of admissible strategies to be

$$\Pi_p := \Big\{\pi = (a, L, G) \,\Big|\, \big(a(t), L(t), G(t)\big) \in \mathcal{F}(t),\ F, G \text{ are non–decreasing and càglàd and } V^{\pi}(x) \geq 0\Big\}.$$

For a given cost factor β, the value function corresponding to the manager's problem is

$$\overline{V}_{\beta}(x) := \sup_{\pi \in \Pi_p} V^{\pi}(x). \tag{2}$$

The results in the related literature suggest that $V_{\beta}(x)$ should be a concave, increasing function, and its first derivative of should be greater or equal than one. This reflects the marginal impact that an additional unit of cash reserves has on the firm's value, and the (possible) decreasing returns to scale. As it is usually done in the optimal control literature, where second–order smoothness of the value function is associated to optimality, we focus on candidate value functions that are of class $\mathcal{C}^2$, in which case we have the following

Proposition 3.1. *Assume the value function defined in Equation* (2) *is concave and twice continuously differentiable. Then the following HJB equation holds in the strong sense:*

$$\max\Big\{\max_{a\in[0,1]}\Big\{\frac{1}{2}\sigma^2 a^2 \overline{V}''_{\beta} + \mu a \overline{V}'_{\beta} - \rho \overline{V}_{\beta}\Big\},\, 1 - \overline{V}'_{\beta}(x),\, \overline{V}'_{\beta}(x) - \beta\Big\} = 0. \tag{3}$$

The Variational Inequality (3) provides necessary conditions for optimality. Furthermore, it follows from Proposition 3.1 that, as long as refinancing only occurs when $R^{\pi}(t) = 0$, the value function satisfies the Neumann condition $\overline{V}'_{\beta}(0) = \beta$. Below we follow a strategy similar to that in [15] and construct a candidate value function V_{β} by initially dividing the domain into three sections: $[0, x_0)$, $[x_0, x_1)$ and $[x_1, \infty)$. The solution to

$$\max_{a\in[0,1]}\Big\{\frac{1}{2}\sigma^2 a^2 V''_{\beta} + \mu a V'_{\beta} - \rho V_{\beta}\Big\} \tag{4}$$

is assumed to be interior on $(0, x_0)$. On (x_0, x_1) we have $a \equiv 1$, whereas V_{β} is affine on $[x_1, \infty)$ and on this domain $V'_{\beta} \equiv 1$. The economic intuition behind this partition is the following: On $(0, x_0)$ the firm's risk exposure is adjusted to decrease the probability of R^{π} becoming negative, which is costly; on (x_0, x_1) the level of cash reserves is too large to require caution, hence the firm is in "full–risk" mode, which is in expectation the most profitable one; level x_1 is the *dividend barrier*, and all cash beyond this point is instantly distributed to the equity holders. Since $V'_{\beta} \equiv 1$ in this region, cash in the firm is worth the same to the shareholders as any dividends would be. As it turns out, only at date $t = 0$ could the cash level be strictly greater than x_1. In this case there would be a lump sum of size $x - x_1$ distributed at $t = 0$. Afterwards, an optimal dividend/revinvestment regime will keep the cash reserves restricted to $[0, x_1]$.

It should be stressed that, although our initial analysis takes the boundary points x_0 and x_1 as given, they will later be chosen as to have a $\mathcal{C}^2$ candidate for the value function. It might occur, however, that for values of β in a small enough neighborhood of one (the radius of such neighborhood depends on μ, σ and ρ), the variable–exposure region disappears. This obeys the fact that very cheap re–financing results in a manger who needs not exercise much caution. We make precise how this situations arises in the following section.

3.1. The full–investment region and the dividend barrier. On (x_0, x_1), and as long as $1 < V'_\beta < \beta$ (which we verify a posteriori), the candidate value function V_β satisfies the second–order, linear ODE

$$\frac{1}{2}\sigma^2 V''_\beta + \mu V'_\beta - \rho V_\beta = 0,$$

whose general solution is

$$V_\beta(x) = b_1 e^{r_1 x} + b_2 e^{r_2 x}, \quad \text{where} \quad r_i := \frac{-\mu + (-1)^i \sqrt{\mu^2 + 2\sigma^2\rho}}{\sigma^2}. \tag{5}$$

For each $x_1 > 0$ we define V_{x_1} as the particular solution to Equation (5) that satisfies $V'_{x_1}(x_1) = 1$ and $V''_{x_1}(x_1) = 0$. It is not complicated to show that

$$V_{x_1}(x) = \frac{1}{r_1 r_2}\Big(\frac{r_2^2}{r_2 - r_1} e^{r_1(x - x_1)} - \frac{r_1^2}{r_2 - r_1} e^{r_2(x - x_1)}\Big). \tag{6}$$

To show that the mapping $x \mapsto V'_{x_1}(x)$ is decreasing for $x < x_1$, we compute

$$V''_{x_1}(x) = \frac{r_1 r_2}{r_2 - r_1}\Big(e^{r_1(x - x_1)} - e^{r_2(x - x_1)}\Big).$$

Observe that $(r_1 r_2)/(r_2 - r_1)$ is negative, so $V''_{x_1}(x)$ is negative whenever $e^{r_1(x-x_1)} - e^{r_2(x-x_1)}$ is positive. The latter is equivalent to $(r_1 - r_2)(x - x_1) > 0$ and, since $r_1 - r_2$ is negative, it is also equivalent to $x < x_1$. In Section 3.3 x_1 is chosen so that V_β is of class $\mathcal{C}^2$, and the condition $V'_{x_1} < \beta$ is satisfied. On $[x_1, \infty)$ the value function has constant derivative equal to one. Since $V_{x_1}(x_1) = \frac{r_2 + r_1}{r_1 r_2} = \frac{\mu}{\rho}$, we have by continuity that

$$V_\beta(x) = (x - x_1) + \frac{\mu}{\rho}, \quad x > x_1.$$

3.2. The variable exposure region. Whenever the firm's level of cash reserves $x \in (0, x_0)$, the manager will choose a risk exposure that depends on x. This is done by solving the problem

$$\max_{a \in [0,1]} \Big\{\frac{1}{2}\sigma^2 a^2 V''_\beta + \mu a V'_\beta - \rho V_\beta\Big\},$$

whose maximizer is interior in this region. The first–order conditions on a are sufficient for optimality, and they yield

$$a(x) = -\frac{\mu}{\sigma^2}\frac{V'_\beta(x)}{V''_\beta(x)}. \tag{7}$$

This in turn transforms Equation (4) into $-\frac{\mu^2}{2\sigma^2}\frac{[V'_\beta(x)]^2}{V''_\beta(x)} = \rho V_\beta(x)$, which has general solution

$$V_\beta(x) = c_1\left(x\frac{\rho}{\gamma} + c_2\right)^\gamma, \text{ where } \quad \gamma := \frac{2\sigma^2\rho}{\mu^2 + 2\sigma^2\rho}.$$

The boundary condition $V'_\beta(0) = \beta$ implies $c_1 = (\beta/\rho)c_2^{1-\gamma}$. For each $c > 0$ we define

$$V_c(x) := \frac{\beta}{\rho}c^{1-\gamma}\left(x\frac{\rho}{\gamma} + c\right)^{\gamma}. \tag{8}$$

In particular, we have that $V_c(0) = \beta c/\rho$. Due to the fact that $\gamma < 1$, V_c is a concave function for all $c > 0$. If we insert Equation (8) into Equation (7) we obtain

$$a(x) = \frac{\mu}{\sigma^2}\frac{x}{1-\gamma} + \frac{2}{\mu}c. \tag{9}$$

Remark 3.2. *Unless $c = 0$ in Equation (9), the firm's risk exposure is not zero when $x = 0$. This obeys the fact that a diminished exposure is not the only tool available to ward off bankruptcy, but additional cash injection may (and will be) used. The amount*

$$\alpha(x) := \frac{\mu}{\sigma^2}\frac{x}{1-\gamma}$$

corresponds to the variable–exposure strategy in the basic model presented in Section 2, where no additional contributions are possible (see Appendix B). In such case the firm's size becomes infinitesimal as x approaches zero, preventing it from hitting the bankruptcy barrier; technically speaking, the optimal strategy implies that the dynamics of the firm's cash reserves follow a reflected geometric Brownian motion. Since $a(x) \geq \alpha(x)$ (and in fact strictly greater if $c > 0$), we may conclude that risk exposure is larger over the variable–exposure region whenever cash re–injection is required to refinance the firm whenever the cash reserves hit zero.

The mapping $x \mapsto a(x)$ is strictly increasing, hence, to find x_0 we solve the equation $a(x) = 1$, which yields

$$x_0 = x_0(c) = \frac{\sigma^2(1-\gamma)}{\mu} - \frac{\gamma}{\rho}c. \tag{10}$$

Unlike in [7] and [15], in our model x_0 is not uniquely defined by the boundary condition at zero. The variable c will be chosen together with x_1 as to guarantee the required regularity of V_β. If as a result of this process, and for μ, σ and ρ given, it is the case that $c = c(\beta)$ satisfies

$$\frac{\sigma^2(1-\gamma)}{\mu} - \frac{\gamma}{\rho}c(\beta) < 0 \Leftrightarrow \frac{\mu}{2} < c(\beta),$$

then the partial–exposure regime disappears.

3.3. Characterizing the optimal strategy. We are seeking a solution of class $\mathcal{C}^2$ to Equation (3). The arguments in Sections 3.1 and 3.2 imply that, as long as $x_0(c) > 0$, the structure of the candidate value function V_β is the following:

$$V_\beta(x) = \begin{cases} V_c(x), & x \in [0, x_0(c)); \\ V_{x_1}(x), & x \in [x_0(c), x_1); \\ (x - x_1) + V_{x_1}(x_1), & x \in [x_1, \infty). \end{cases} \tag{11}$$

The smooth–pasting conditions require us to find c^* and x_1^* that solve the constrained, non–linear system of equations

$$V_{c^*}(x_0(c^*)) = V_{x_1^*}(x_0(c^*)), \quad V'_{c^*}(x_0(c^*)) = V'_{x_1^*}(x_0(c^*)), \quad c^*, x_1^* \geq 0. \tag{12}$$

Although in general such a system needs not have a (unique) solution, an issue we shall encounter in Section 4, such is not the case for System (12). The following theorem is one of the main result in this section, and its proof allows us to draw several conclusions regarding the effects of the cost factor on the manager's decisions.

Theorem 3.3. *For any triple $(\mu,\sigma,\rho)>0$ and any $\beta>1$ System* (12) *has a unique solution.*

If for a triple $(\mu,\sigma,\rho)>0$ and $\beta>1$ the solution to System (12) yields $x_0(\beta):=x_0(c^*(\beta))<0$, then V_β is characterized via

$$V_\beta(x) = \begin{cases} V_{x_1^*}(x), & x\in[0,x_1^*); \\ (x-x_1^*)+V_{x_1^*}(x_1^*), & x\in[x_1,\infty), \end{cases}$$

where x_1^* solves

$$r_2e^{-r_1x_1^*}-r_1e^{-r_1x_1^*}=(r_2-r_1)\beta. \tag{13}$$

The mapping $x_1\mapsto r_2e^{-r_1x_1}-r_1e^{-r_1x_1}$ is strictly increasing for $x_1>0$, and it attains the value r_2-r_1 for $x_1=0$; therefore, Equation (13) has a unique positive solution for $\beta>1$. Notice that in this case $V_{x_1^*}(0)>0$, which follows from the fact that $r_2^2e^{-r_1x_1^*}<r_1^2e^{-r_2x_1^*}$.

In the analysis below we let μ,σ and ρ be fixed and discuss the impact of a varying β. An increasing level of β should depress the corresponding value function. In particular one would expect $V_\beta(0)$, which equals $\beta c(\beta)/\rho$, to be decreasing in β. The latter would require that $c(\beta)$ tends to 0 as β tends to ∞, which would also affect the value of the lower boundary of the full–exposure region. Namely, $x_0(\beta)$ would converge to $\sigma^2(1-\gamma)/\mu$ and $a(x)$ would converge to $\alpha(x)$ as c goes to 0. This quantities are precisely the boundary of the variable and full–exposure regions in the basic model, as well as the corresponding level of exposure in the variable–exposure region. At the limit there should in fact be no additional contributions, given that their marginal cost becomes infinite. In this case we would have a "limit value function" that satisfies $V_\infty(0)=0$ and $V'_\infty(0)=\infty$, and for which the boundary of the variable–investment region, as well as the level of investment, coincides with the values in the basic model. We prove in Proposition 3.5 that the value function $U(\cdot)$ bounds $V_\beta(\cdot)$ from below, and that $\|U-V_\beta\|_\infty$ converges to 0 as β goes to ∞. On the other hand, as β tends to 1 the value function $V_\beta(\cdot)$ becomes less concave, which is reflected by an increasing value of $c(\beta)$. Eventually $c(\beta)$ becomes larger than $\mu/2$ for $\beta<1$, and the variable–exposure region vanishes. At the limit the value function $V_\beta(x)$ becomes linear. The limiting function is

$$W(x):=x+\frac{\mu}{\rho},$$

which corresponds to the case where an exceptional dividend of size x is distributed at time zero, and the future value of the firm corresponds to the discounted average profitability of the fully–exposed strategy. We present in Figure 1 the plots of the value functions corresponding to the parameters $\mu=1.3,\rho=1.2,\sigma=1$ and different values of β. For this particular example, x_0 becomes zero for $\beta\approx 1.25093$.

The proof of Theorem 3.3 also yields the following

Corollary 3.4. *For $(\mu,\sigma,\rho)>0$ fixed, there exists strictly positive constants $A_5(\mu,\sigma,\rho)$ and $A_6(\mu,\sigma,\rho)$ such that β parameterizes c in the following way: $c(\beta)=A_5(\mu,\sigma,\rho)\beta^{1/(\gamma-1)}$. If $c(\beta)<\mu/2$ then it holds that $x_1(\beta)=x_0(\beta)+A_6(\mu,\sigma,\rho)$ and the exposure strategy is given by*

$$a_\beta(x):=\frac{\mu}{\sigma^2}\frac{x}{1-\gamma}+\frac{2}{\mu}c(\beta).$$

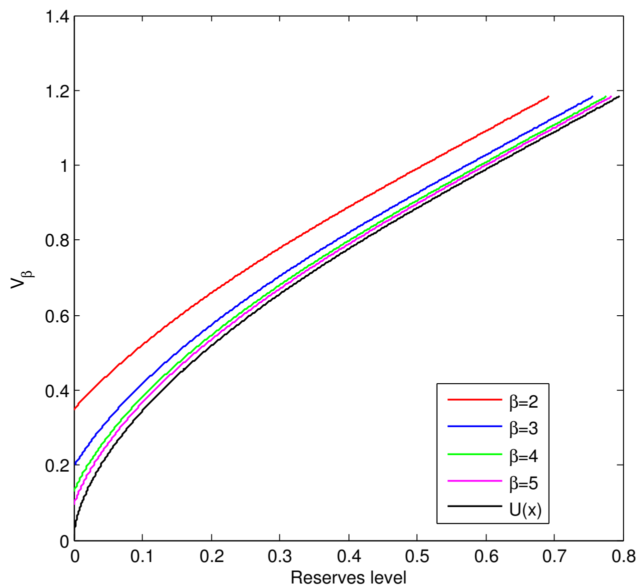


FIGURE 1. Comparing different V_β's with $\mu = 1.3, \rho = 1.2, \sigma = 1$.

Furthermore

$$V_{c(\beta)}(x_0(\beta)) = V_{x_1(\beta)}(x_0(\beta)) = A_5(\mu,\sigma,\rho)^{1-\gamma}\frac{1}{\rho}\Big(\frac{\mu}{2}\Big)^{\gamma}$$

and

$$V'_{c(\beta)}(x_0(\beta)) = V'_{x_1(\beta)}(x_0(\beta)) = A_5(\mu,\sigma,\rho)^{1-\gamma}\Big(\frac{\mu}{2}\Big)^{\gamma-1}$$

are independent of β.

Corollary 3.4 confirms our intuition that both $c(\beta)$ and $V_\beta(0) = \rho^{-1}A_5\beta^{\gamma/(\gamma-1)}$ tend to zero as β tends to infinity. Moreover, $A_6(\mu,\sigma,\rho)$ coincides with $x_1 - x_0$ in the no–reinvestment case, which implies that the presence of the re–investment opportunity shifts the full–exposure region to the left. This leads us to the following

Proposition 3.5. *Let* $(\mu,\sigma,\rho) > 0$ *be given. Then*

(i) $U(x) < V_\beta(x)$ *for all* $x \geq 0$ *and* $\beta < 1$ *and* $\lim_{\beta\to\infty}\|U - V_\beta\|_\infty = 0$.
(ii) $V_\beta(x) < W(x)$ *for all* $x \geq 0$ *and* $\beta < 1$ *and* $\lim_{\beta\to 1}\|W - V_\beta\|_\infty = 0$.

As a consequence of Proposition 3.5 we have that removing the restriction of mandatory reinvestment would not alter the manager's decisions. In other words, regardless of how high the marginal cost of reinvestment is, the shareholders' interests are best served by choosing such option. As long as $c(\beta) < \mu/2$, the value at which the risk exposure changes from partial to full is independent of β, since $V_{c(\beta)}\big(x_0(\beta)\big)$ and $V'_{c(\beta)}\big(x_0(\beta)\big)$ are constant. The level of cash reserves at which this switch occurs, though, becomes smaller with β. Furthermore, the value at which dividends are distributed is also independent of β.

3.4. A verification result and implementation of the optimal strategies. For given parameters β, μ, σ, and ρ, if we insert $c(\beta)$ and $x_1(\beta)$ obtained in Corollary 3.4 into the function defined in Expression (11) we obtain a candidate solution V_β to Equation (2). In this section we provide a verification result, and make the processes $L(t)$ and $G(t)$ explicit.

The Neumann conditions at $x = 0$ and $x = x_1(\beta)$ suggest that a localization strategy will be optimal. Let us consider a Skorohod problem on $[0, x_1(\beta)]$ defined as follows: For $a(x)$ defined in Equation (9) let the processes (R^*, L^*, G^*) be a solution to

$$R^*(t) = x + \int_0^t \mu a(R^*(s))ds + \int_0^t \sigma a(R^*(s))dW(s) - L^*(t) + G^*(t), \tag{14}$$

$$0 \leq R^*(t) \leq x_1(\beta),\ t \geq 0, \tag{15}$$

$$\int_0^\infty \mathbb{1}_{\{R^*(t)<x_1(\beta)\}} dL^*(t) = \int_0^\infty \mathbb{1}_{\{R^*(t)>0\}} dG^*(t) = 0, \tag{16}$$

where $\mathbb{1}_{\{\cdot\}}$ is the zero–one indicator function. A comprehensive treatise on such reflection problems can be found in [3] and [11]. The processes $L^*(t)$ and $G^*(t)$ are the local times of $R^*(t)$ at levels $x_1(\beta)$ and 0 respectively (See [14] for a thorough exposition of Brownian local times). Their effect on the dynamics of $R^*(t)$ is to reflect the latter in order to constrain it to $[0, x_1(\beta)]$. From Equation (16) we see that the mass of the measures $dL^*(t)$ and $dG^*(t)$ is carried respectively by the sets $\{R^*(t) = x_1(\beta)\}$ and $\{R^*(t) = 0\}$, thus $L^*(t)$ and $G^*(t)$ are inactive whenever $R^*(t) \in (0, x_1(\beta))$. We prove in Theorem 3.6 that the strategy $\pi^* := (a^*, L^*, G^*)$, with $a^*(t) := a(R^*(t))$ solves the investor's problem in Expression (2).

Notice that, by construction, V_β is of class $\mathcal{C}^2$ and it satisfies the HJB Equation (3). We prove that V_β is the value function defined in Expression (2) in the following

Theorem 3.6. *Let $\beta \in [1, \infty)$ and $\mu, \sigma, \rho > 0$. Let $c(\beta)$ and $x_1(\beta)$ be as in Corollary 3.4 and V_β the corresponding candidate value function. Let $a^*(x) = a_\beta(x)$ and $\pi^* = (a^*, L^*, G^*)$, then for all $x \geq 0$*

$$\overline{V}_\beta(x) = V_\beta(x) = V^{\pi^*}(x).$$

We have seen that, for any β greater than one and conditional on cash re–injection being only possible at level $x = 0$, costly refinancing is always a dominant strategy. To finalize this section we show that the constraint on the re–injection level is moot, since refinancing at any other level is suboptimal.

Proposition 3.7. *Let $\beta \in [1, \infty)$ and $\mu, \sigma, \rho > 0$. Let $x > 0$ and consider $\hat{x} \in (0, x)$. Denote by $\hat{V}_\beta(x)$ the function defined in Expression (2) when re–investment has to be made at level $\hat{x}$. Then for all $x \geq \hat{x}$ it holds that $\hat{V}_\beta(x) < V_\beta(x)$.*

We have that, in fact, for $x \in [\hat{x}, \infty)$, $\hat{V}_\beta(x) = V_\beta(x - \hat{x})$, thus $\hat{V}'_\beta(\hat{x}) = \beta$ and the constraint $\hat{V}'_\beta \geq \beta$ implies that the only possible continuous continuation of $\hat{V}_\beta$ to $[0, \hat{x})$ is an affine one, i.e.

$$\hat{V}_\beta(x) = \beta(x - \hat{x}) + \frac{A_5}{\rho}\beta^{\gamma/(\gamma-1)},$$

where $(A_5/\rho)\beta^{\gamma/(\gamma-1)} = \hat{V}_\beta(\hat{x})$ (see the proof of Proposition 3.7). This produces a non–$\mathcal{C}^2$ pasting at $\hat{x}$. The situation $x < \hat{x}$ can therefore only occur at date $t = 0$, when an exceptional, lumpy re–injection of cash of amount $\hat{x} - x$ would be required. The latter as long as

$(A_5/\rho)\beta^{\gamma/(\gamma-1)} > \hat{x}$, since otherwise the firm would have a negative NPV. In other words, if re–injection at level $\hat{x}$ were imposed on the firm, the firm would only be initiated for small values of $\hat{x}$ and/or β.

4. Fixed Costs of Refinancing

In this section we assume there is a non–negative, fixed cost of reinvestment, which we denote by κ. In other words, to re–inject x units of cash into the firm, the shareholders must supply $x+\kappa$. We prove that, among strategies that require cash re–injection, doing so at level zero is dominant. The latter suggests, within our stationary context, that there should be a strictly positive level $\hat{x}$ of funds to be contributed, should the manager chose to do so, if the cash reserves reach zero. Naturally, if κ were too high, cash re–injection would be unfeasible. As a consequence, and in contrast with Section 3, we do not initially assume bankruptcy is always avoided. Notice that a lumpy refinancing strategy will result in a cash–reserves process that is no longer reflected at level zero. Instead, the process will exhibit jumps of magnitude $\hat{x}$ at the bankruptcy level. This discontinuous behavior of the state and control variables situates the results contained in this section within the *impulse control literature* (see, for example [10]).

4.1. The (possible) contributions' structure and the value function. We start by proving that it is optimal to postpone any re–injection of cash until the level $x=0$ has been reached. Due to the presence of the strictly positive fixed cost κ, it would be unfeasible for refinancing to take place outside a discrete set of points. Otherwise, costs would become infinite in finite time. Accordingly, we use the following notion of admissibility for an exposure–dividends–refinancing triple:

Definition 4.1. *Let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of strictly increasing $\{\mathcal{F}_t\}_{t\geq 0}$–stopping times. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of non–negative random variables such that $u_i \in \mathcal{F}_{\tau_i}$. Define*

$$J(t) := \sum_{i=1}^{\infty} x_i \mathbb{1}_{\{t\geq\tau_i\}} \quad \textit{and} \quad J_\kappa(t) := \sum_{i=1}^{\infty}(x_i+\kappa)\mathbb{1}_{\{t\geq\tau_i\}}.$$

Further, let $L(t)$ and $a(t)$ be a cumulative consumption process and an exposure strategy, respectively. For a given initial level of cash reserves x, we say that the triple (a, L, J) is admissible if the following conditions hold:

1. *The process $R(t)$, whose dynamics are given by*
$$dR(t) = a(t)(\mu dt + \sigma dW(t)) - dL(t) + dJ(t), \quad R(0) = x$$
is a.s. non–negative.
2. *The value of the firm is non–negative, i.e.*
$$V(x) = \mathbb{E}\Big[\int_0^{\infty} e^{-\rho t}(dL(t) - dJ_\kappa(t))\Big] \geq 0.$$

Consider now an initial level of cash reserves x, an admissible triple (a, L, J) and the corresponding value $V(x)$. We may assume without loss of generality that $R(\tau_1) > 0$. Let us define the auxiliary processes

$$\tilde{J}(t) := \sum_{i=2}^{\infty} x_i \mathbb{1}_{\{t\geq\tau_i\}}$$

and the corresponding $\tilde{R}(t)$, together with the stopping time $\tau := \inf\{t > 0 \mid \tilde{R}(t) \le 0\}$. By construction $\tau > \tau_1$. If we define $\overline{J}(t) := \tilde{J}(t) + x_1 1\!\!1_{\{t \ge \tau\}}$ we have that the triple $(a, L, \overline{J})$ is admissible and the corresponding value $U(x)$ satisfies $U(x) > V(x)$. This last assertion follows from the facts that $-e^{-\rho\tau}(x_1 + \kappa) < e^{-\rho\tau_1}(x_1 + \kappa)$, and that for any $t \ge \tau_2$ it holds that $R(t) = \tilde{R}(t)$. It is then optimal to postpone the first time of issuance to the first time the cash–reserves process hits zero, and an inductive argument implies that such is the case for all re–injection times. Furthermore, the stationarity of our model also implies that there exists $\hat{x} \ge 0$ such that $x_i \equiv \hat{x}$ for all $i \ge 1$.

We are now in a position to define the family of stopping times when cash could be re–injected into the firm. For a given exposure–consumption pair (a, L) and initial cash reserves x, let us define the sequence $\{\lambda_i\}$ as follows:

$\lambda_1 := \inf\Big\{t > 0 \Big| R_1^{a,L}(t) \le 0\Big\}$, where $R_1^{a,L}$ solves $dR_1^{a,L}(t) = a(t)(\mu dt + \sigma dW(t)) - dL(t),\ R_1^{a,L}(0) = x$;

for $i > 1$ we define $\lambda_i := \inf\big\{t > \lambda_{i-1} \big| R_i^{a,L}(t) \le 0\big\}$, where $R_i^{a,L}$ solves

$$dR_i^{a,L}(t) = a(t)(\mu dt + \sigma dW(t)) - dL(t),\ R_i^{a,L}(\lambda_{i-1}) = \hat{x}.$$

Let $\{J(t)\}$ be the (jump) process defined via

$$J(t) = \begin{cases} 0, & t \in [0, \lambda_1); \\ i - 1, & x \in [\lambda_{i-1}, \lambda_i). \end{cases} \tag{17}$$

Then the cumulative reinvestment process corresponding to a re–injection level $\hat{x}$ and cost κ is $G(t) = (\hat{x} + \kappa)J(t)$. If the manager chooses the triple $\pi = (a, L, \hat{x})$, then the dynamics of the cash–reserves process are

$$dR^\pi(t) = a(t)(\mu dt + \sigma dW(t)) - dL(t) + \hat{x}dJ(t), \quad R^\pi(0) = x.$$

The corresponding value of the firm is

$$V^\pi(x) = \mathbb{E}\Big[\int_0^\infty e^{-\rho t}(dL(t) - dG(t))\Big] = \mathbb{E}\Big[\int_0^\infty e^{-\rho t}(dL(t) - (\hat{x} + \kappa)dJ(t))\Big].$$

An increase in $\hat{x}$ increases the probability that consumption takes place, but it also costly. We redefine the set of admissible strategies to be

$$\Pi_f := \Big\{\pi = (a, L, \hat{x}) \Big| \big(a(t), L(t)\big) \in \mathcal{F}(t),\ L \text{ is non–decreasing and càdlàg},\ \hat{x} \ge 0 \text{ and } V^\pi(x) \ge 0\Big\}.$$

As before, the manager's objective is to find

$$\overline{V}_\kappa(x) := \sup_{\pi \in \Pi_f} V^\pi(x). \tag{18}$$

Proposition 4.2. *Assume the value function defined in Equation* (18) *is twice continuously differentiable. Let $\kappa \ge 0$ be the fixed cost of reinvestment. Then the following HJB equation holds in the strong sense:*

$$\max\left\{\max_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2 a^2 \overline{V}''_\kappa + \mu a \overline{V}'_\kappa - \rho \overline{V}_\kappa\right\},\ 1 - \overline{V}'_\kappa(x),\ H(x, \overline{V}_\kappa) - \overline{V}_\kappa(x)\right\} = 0, \tag{19}$$

where

$$H(x, \phi) := \sup_{u \ge 0}\big\{\phi(x + u) - u - \kappa\big\}.$$

Observe that Proposition 4.2 implies the boundary condition at $x = 0$ must also be determined endogenously. Namely, any candidate value function V_κ must satisfy the non–local boundary condition $V_\kappa(0) = H(0, V_\kappa)$. If we compute the first–order conditions for $\sup_{u\geq 0} \big\{V_\kappa(0+u) - u - \kappa\big\}$, we obtain that u must satisfy $V_\kappa'(u) = 1$. This suggest that we will again have a two–regimes setting of partial and full exposure, and distribution of dividends will occur only when $V_\kappa'(x) = 1$. We shall again denote this optimal level of cash reserves by x_1. Given that we now face fixed refinancing costs, the condition $V_\kappa'(u) = 1$ confirms the intuition that if refinancing is required, then x_1 should be the optimal level of new contributions. This, naturally, as long as it it not the case that too–high fixed costs make refinancing unprofitable. In other words, to allow for strategic default the boundary condition should be rewritten as

$$V_\kappa(0) = \max\big\{V_\kappa(x_1) - x_1 - \kappa, 0\big\}.$$

Below we follow a similar strategy as in Section 3 and analyze the investor's decisions on the regions $(0, x_0)$ and (x_0, x_1) to construct a candidate value function V_κ.

4.2. The investment regions and the distribution barrier. There are two main departures, in terms of the differential equations, between the fixed and the proportional–costs cases. The first one corresponds to the boundary condition at $x = 0$. The second one to the determination of the (free) boundary that separates the variable and full–exposure regions. Bellow we analyze the impact of a varying level of κ for fixed μ, ρ and σ. In the case of the full–exposure region (x_0, x_1), the results of Section 3.1 still apply, thus any candidate value function has the following form:

$$V_{x_1}(x) = \frac{1}{r_1 r_2}\Big(\frac{r_2^2}{r_2 - r_1}e^{r_1(x-x_1)} - \frac{r_1^2}{r_2 - r_1}e^{r_2(x-x_1)}\Big).$$

For $x > x_1$ the candidate value function is again an affine function of the cash–reserves level x :

$$V_\kappa(x) = (x - x_1) + \frac{r_2 + r_1}{r_1 r_2} = (x - x_1) + \frac{\mu}{\rho}.$$

Borrowing from Section 3.2, we have that on $(0, x_0)$ the candidate value function is of the form

$$V_\kappa(x) = c_1\left(x\frac{\rho}{\gamma} + c_2\right)^\gamma,$$

where again $\gamma = \rho/\big(\frac{\mu^2}{2\sigma^2} + \rho\big)$. Assume that for x_1 given, $V(x_1) - x_1 - \kappa > 0$. Then the boundary condition $V(0) = V(x_1) - x_1 - \kappa$ implies

$$c_2 = \Big(\frac{1}{c_1}\big[\frac{\mu}{\rho} - x_1 - \kappa\big]\Big)^{1/\gamma}.$$

For each $c > 0$ we have

$$V_{c,x_1}(x) := c\left(x\frac{\rho}{\gamma} + \mathbf{s}(c, x_1)\right)^\gamma, \quad \text{where} \quad \mathbf{s}(c, x_1) := \Big(\frac{1}{c}\big[\frac{\mu}{\rho} - x_1 - \kappa\big]\Big)^{1/\gamma}. \tag{20}$$

On the variable risk exposure region $(0, x_0)$, the equation $a(x) = -\frac{\mu}{\sigma^2}\frac{V_\kappa'(x)}{V_\kappa''(x)}$ still holds, which together with Equation (20) yields

$$a(x) = \frac{2}{\mu}\big(x\frac{\rho}{\gamma} + \mathbf{s}(c, x_1)\big). \tag{21}$$

Once again we determine x_0 as the solution to the equation $a(x) = 1$, which in this case results in

$$x_0 = x_0(c, x_1) = \frac{\gamma}{\rho}\Big(\frac{\mu}{2} - \mathbf{s}(c, x_1)\Big) = \frac{\sigma^2(1-\gamma)}{\mu} - \frac{\gamma}{\rho}\mathbf{s}(c, x_1). \tag{22}$$

The condition that guarantees a non–void partial–exposure region is $\mathbf{s}(c, x_1) < \mu/2$.

Proposition 4.3. *Let $c^*(\kappa)$ and $x_1^*(\kappa)$ be parameters that yield a $\mathcal{C}^2$ candidate value function. There exists a threshold $\underline{\kappa} > 0$ (which depends on μ, σ and ρ) such that $\mathbf{s}(c(\kappa), x_1(\kappa)) > \mu/2$ for $\kappa < \underline{\kappa}$.*

If it were the case that $\kappa < \overline{\kappa}$, then the manager's problem would reduce to deciding when to distribute dividends. The dividend barrier $x_1^*(\kappa)$ solves the equation

$$\frac{1}{r_1 r_2}\Big(\frac{r_2^2}{r_2 - r_1}e^{-r_1 x_1^*(\kappa)} - \frac{r_1^2}{r_2 - r_1}e^{-r_2 x_1^*(\kappa)}\Big) = \frac{\mu}{\rho} - x_1^*(\kappa) - \kappa. \tag{23}$$

The following lemma follows from Proposition 4.3 and Equation (23)

Lemma 4.4. *For any μ, ρ and σ, it holds that $\lim_{\kappa\to 0} x_1^*(\kappa) = 0$.*

In particular, if $\kappa = 0$ then $x_1 = 0$ solves Equation (23). In other words, costless reinvestment results in the first–best solution, i.e. $V_\kappa(x) = W(x)$, which coincides with the proportional–cost case when $\beta = 1$.

Remark 4.5. *In contrast with the proportional–costs scenario, here $x_1 - x_0$ will not be constant. However, we are still able to express $x_0(c, x_1)$ as the difference between the x_0 of the basic model and the non–negative term $\frac{\gamma}{\rho}\mathbf{s}(c, x_1)$. Therefore, the firm enters the full–exposure regime earlier (or at least not later) with the option of refinancing subject to a fixed–cost. Furthermore, this non–constant size of the full–exposure region implies that, in general, for μ, σ and ρ given it is not possible to find $\beta > 1$ and $\kappa > 0$ such that $V_\beta(x) = V_\kappa(x)$ for all $x \geq 0$.*

Let us revisit the condition $V_\kappa(0) = \max\{V_\kappa(x_1) - x_1 - \kappa, 0\} = \max\{\frac{\mu}{\rho} - x_1 - \kappa, 0\}$. If $\kappa > \mu/\rho$ then the cost of refinancing, relative to the firm's discounted, expected cashflows, would be too high. As a consequence $V(0) = 0$ and we would find ourselves back in the basic model. The condition $\kappa < \mu/\rho$ is necessary, but not sufficient, for refinancing to be profitable. Below we analyze the effects that different levels of κ have have on the boundary conditions.

4.3. Characterizing the optimal strategy. We seek smooth pasting of the candidate value function $\tilde{V}_\kappa(x)$, which is defined, for a given $0 < \kappa < \overline{\kappa}$, as:

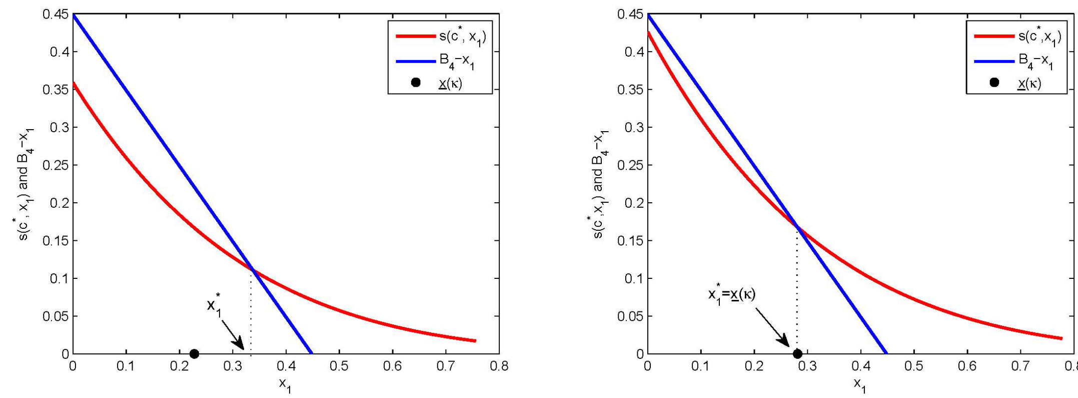
$$V_\kappa(x) := \begin{cases} V_{c^*, x_1^*}(x), & x \in [0, x_0(c^*, x_1^*)); \\ V_{x_1^*}(x), & x \in [x_0(c^*, x_1^*), x_1^*); \\ (x - x_1^*) + \mu/\rho, & x \in [x_1^*, \infty), \end{cases} \tag{24}$$

where c^* and x_1^* solve the constrained, non–linear system of equations

$$\begin{aligned} &V_{c^*, x_1^*}(x_0(c^*, x_1^*)) = V_{x_1^*}(x_0(c^*, x_1^*)),\ V'_{c^*, x_1^*}(x_0(c^*)) = V'_{x_1^*}(x_0(c^*, x_1^*)), \\ &0 \leq x_1^* \leq \frac{\mu}{\rho} - \kappa;\ c^* \geq 0. \end{aligned} \tag{25}$$

We present necessary conditions for optimality of V_κ in the following

Proposition 4.6. *Let $(\mu, \sigma, \rho) > 0$, set $A := \sqrt{\mu^2 + 2\sigma^2\rho}$ and assume $0 < \kappa < \mu/\rho$. Any solution to System (25) satisfies*



(a) $\kappa = 0.1$ (b) $\kappa \approx 0.028$

- $c^* = \dfrac{2}{A\sqrt{-r_1 r_2}}\Big(-\dfrac{r_2}{r_1}\Big)^{-\frac{\mu}{2A}}\Big(\dfrac{\mu}{2}\Big)^{1-\gamma}$.
- *The value of* $x_1^* = x_1^*(\kappa)$ *is implicitly determined by the expression*

$$\frac{\gamma}{\rho}\mathbf{s}(c^*, x_1^*) = B_4(\mu, \sigma, \rho) - x_1^*, \tag{26}$$

where $B_4(\mu, \sigma, \rho) := -\dfrac{\sigma^2}{2A}\log\Big(-\dfrac{r_2}{r_1}\Big) + \dfrac{\gamma\mu}{2\rho}$.

Observe that the value of c^* does not depend on κ. Equation (26) provides us with the precise value $\overline{\kappa}$ of the fixed cost for which the manager is indifferent between refinancing or not. This occurs when the graphs of the mappings $x \mapsto \mathbf{s}(c^*, x)$ and $x \mapsto B_4(\mu, \sigma, \rho) - x$ intersect at $x = B_4(\mu, \sigma, \rho)$. In other words, $\overline{\kappa}$ is such that $\mathbf{s}(c^*, B_4) = 0$:

$$\overline{\kappa} = \frac{\mu}{\rho}\Big(1 - \frac{\gamma}{2}\Big) + \frac{\sigma^2}{2A}\log\Big(-\frac{r_2}{r_1}\Big).$$

For $\kappa = \overline{\kappa}$ the value of x that satisfies $\mathbf{s}(c^*, x) = \mu/2$, and therefore determines whether the partial–exposure region is void or not, is

$$\underline{x}(\kappa) = \underline{x}(\overline{\kappa}) = B_4(\mu, \sigma, \rho) - c^*\Big(\frac{\mu}{2}\Big)^{\gamma} < x_1^*(\overline{\kappa}).$$

In fact, $\underline{x}(\overline{\kappa})$ and $x_1^*(\overline{\kappa})$ coincide with x_0 and x_1 in the basic model. As κ becomes smaller than $\overline{\kappa}$, $x_1^*(\kappa)$ becomes smaller than $B_4(\mu, \sigma, \rho)$. On the contrary, $\underline{x}(\kappa)$ grows as κ decreases. As long $\kappa \in (\underline{\kappa}, \overline{\kappa})$, i.e. if $x_0(c^*, x_1^*) > 0$ the variable–exposure regime is given by

$$a_\kappa(x) := \frac{2}{\mu}\big(x\frac{\rho}{\gamma} + \mathbf{s}(c^*, x_1^*(\kappa))\big).$$

Eventually $\kappa = \underline{\kappa}$ and then $\underline{x}(\underline{\kappa}) = x_1^*(\underline{\kappa})$. For any smaller fixed–cost of reinvestment, the variable–exposure region vanishes and $x_1^*(\kappa)$ is determined by solving Equation (23). In other words, if $\kappa \leq \underline{\kappa}$ then $a_\kappa(x) \equiv 1$. We illustrate the above description in Figure 4.3, where we have set $\mu = \sigma = 0.2$ and $\rho = 0.1$.

4.4. Implementation of the optimal strategies. Equation (24) yields a candidate value function whenever $0 < \kappa < \overline{\kappa}$. Associated to such candidate we have the following choice of contributions/dividents strategies: Let $\{L^*\}$ be the local time at level $x_1^*(\kappa)$ of the process X corresponding to the strong solution to

$$dX(t) = a^*(t)(\mu dt + \sigma dW(t)), \quad X(0) = x.$$

As in Section 3, the continuous process $\{L^*\}$ represents the cumulative dividends. From the arguments in Section 4.1 it follows that the only possibly optimal choice for the cumulative contributions process G is given by

$$G^*(t) = (x_1^*(\kappa) + \kappa)J^*(t),$$

where the pure–jump process $\{J^*\}$ is as defined in Equation (17) and the hitting times $\{\lambda_i\}$ are determined by the processes $\left\{R_i^{a^*,L^*}\right\}$. We define $\pi^* := \left(a_\kappa, L^*, x_1^*(\kappa)\right)$ and must now show that $V_\kappa(x) = V^{\pi^*}(x)$ coincides with the value function, which we prove in the following

Theorem 4.7. *Let $\kappa > 0$ and $\mu, \sigma, \rho > 0$. Let c^* and $x_1(\kappa)$ be as in Proposition4.6 and let V_κ be the corresponding candidate value function. Let π^* be as defined above. Then for all $x \geq 0$*

$$\overline{V}_\kappa(x) = V_\kappa(x) = V^{\pi^*}(x).$$

5. Conclusions

We have employed Singular Stochastic Control and Impulse Control techniques to extend the models presented in [7], [4] and [12]. In a setting where a firm's manager continuously adjusts the firm's exposure to a risky production technology, we have studied the impact that costly re–investment possibilities have on the manager's optimal behavior. We have obtained expressions that characterize, and in the proportional–costs case analytically describe, the partial and full exposure regions, as well as the consumption boundary. This has allowed us to provide detailed descriptions of the value functions as re–investment costs vary, such as conditions that lead to the disappearance of the partial exposure region for low cost levels. Furthermore, these expressions provide a clear picture of the quantitative difference between proportional– and fixed–costs models. He have left for future research the interesting scenario that includes a stochastic short rate, an which wold require, in our opinion, a variable investment level in the risky technology in nominal terms.

Appendix A. Proofs

Proof of Proposition 3.1. Within this proof, since there is no risk of confusion, we drop the subindex β. We first consider the conditions on V'. By definition, for any $h, y > 0$ there exists a strategy π_y such that $V^{\pi_y}(y) \geq V(y) - h^2$. Let $0 < h < x$ and construct a strategy π by setting $L_t^\pi = h + L_t^{\pi_{x-h}}$, $G_t^\pi = G_t^{\pi_{x-h}}$ and $a^\pi(t) = a^{\pi_{x-h}}(t)$. Then

$$V(x) \geq V^\pi(x) = h + V^{\pi_{x-h}}(x-h) \geq h + V(x-h) - h^2,$$

which is equivalent to

$$\frac{V(x) - V(x-h)}{h} \geq 1 - h.$$

By the differentiability of V, we may let h go to zero and conclude that $V'(x) \geq 1$. Now set $G_t^\pi = h + L_t^{\pi_{x+h}}$, $L_t^\pi = L_t^{\pi_{x+h}}$ and $a^\pi(t) = a^{\pi_{x+h}}(t)$. Then

$$V(x) \geq V^\pi(x) = V^{\pi_{x+h}}(x+h) - \beta h \geq V(x+h) - \beta h - h^2,$$

therefore

$$\frac{V(x+h)-V(x)}{h} \leq \beta + h.$$

The above implies that $V'(x) \leq \beta$. Next, fix a strategy $\pi \in \Pi_p$ with corresponding cash–reserves process $R^\pi(t)$ ($R^\pi(0) = x$) and apply Itô's formula to $f(t,x) = e^{-\rho t}V(x)$:

$$\begin{aligned} e^{-\rho t}V(R^\pi(t)) = V(x) &+ \int_0^t e^{-\rho s}\big(\mu a(s)V'(R^\pi(s)) - \rho V(R^\pi(s))\big)ds \\ &+ \int_0^t e^{-\rho s}V''(R^\pi(s))d[R^\pi,R^\pi](s) \\ &+ \int_0^t e^{-\rho s}V'(R^\pi(s))dW(s) - \int_0^t e^{-\rho s}V'(R^\pi(s))\big(dL(s)-dG(s)\big) \\ &+ \sum_{s\in\Gamma_1\cup\Gamma_2} e^{-\rho s}\big(V(R^\pi(s)) - V(R^\pi(s_-)) - V'(R^\pi(s))(R^\pi(s) - R^\pi(s_-))\big), \end{aligned} \tag{27}$$

where Γ_1 is the set of discontinuities of L and Γ_2 of G. Since L and G are of bounded variation we have that

$$d[R^\pi,R^\pi](s) = \frac{\sigma^2 a^2(s)}{2}ds.$$

Thus, Equation (27) becomes

$$\begin{aligned} e^{-\rho t}V(R^\pi(t)) = V(x) &+ \int_0^t e^{-\rho s}\mathcal{L}^a(V(R^\pi(s)))ds \\ &+ \int_0^t e^{-\rho s}V'(R^\pi(s))dW(s) - \int_0^t e^{-\rho s}V'(R^\pi(s))\big(dL(s)-dG(s)\big) \\ &+ \sum_{s\in\Gamma_1\cup\Gamma_2} e^{-\rho s}\big(V(R^\pi(s)) - V(R^\pi(s_-)) - V'(R^\pi(s))(R^\pi(s) - R^\pi(s_-))\big), \end{aligned} \tag{28}$$

where $\mathcal{L}^a$ is the infinitesimal generator of the diffusion part of R^π. We now take expectations on both sides. The bounds on V' imply that the Itô integral vanishes and that

$$\mathbb{E}\big[e^{-\rho t}V(R^\pi(t))\big] \leq V(x) - \mathbb{E}\Big[\int_0^t e^{-\rho s}V'(R^\pi(s))\big(dL(s)-dG(s)\big)\Big].$$

Therefore

$$\begin{aligned} 0 \geq\ & \mathbb{E}\Big[\int_0^t e^{-\rho s}\big(-\rho V(R^\pi(s)) + \mu a V'(R^\pi(s)) + \frac{1}{2}\sigma^2 a^2 V''(R^\pi(s))\big)ds\Big] \\ &+ \mathbb{E}\Big[\sum_{s\in\Gamma_1\cup\Gamma_2} e^{-\rho s}\big(V(R^\pi(s)) - V(R^\pi(s_-)) - V'(R^\pi(s))(R^\pi(s) - R^\pi(s_-))\big)\Big]. \end{aligned} \tag{29}$$

By the Mean Value Theorem there exists $r^* \in (R^\pi(s_-), R^\pi(s))$ such that

$$V(R^\pi(s)) - V(R^\pi(s_-)) = V'(r^*)(R^\pi(s) - R^\pi(s_-)).$$

Therefore

$$V(R^\pi(s)) - V(R^\pi(s_-)) - V'(R^\pi(s))\big(R^\pi(s) - R^\pi(s_-)\big) = \big(V'(r^*) - V'(R^\pi(s))\big)\big(R^\pi(s) - R^\pi(s_-)\big)$$

and by concavity of V the second summand of the righthand side of Expression (29) is positive. This yields

$$0 \geq \mathbb{E}\Big[\int_0^t e^{-\rho s}\big(-\rho V(R^\pi(s)) + \mu a V'(R^\pi(s)) + \frac{1}{2}\sigma^2 a^2 V''(R^\pi(s))\big)ds\Big].$$

Multiply both sides of the equation above times $1/t$ and take limit as $t \to 0$. Since the choice of a was arbitrary, this results in

$$0 \geq \max_{a\in[0,1]}\Big\{\frac{1}{2}\sigma^2 a^2 V''(x) + \mu a V'(x) - \rho V(x)\Big\}$$

To prove that one of the inequalities is always tight, we resort to the dynamic programming principle and write for $t > 0$

$$V(x) = \max_{\pi\in\Pi_p}\mathbb{E}\Big[\int_0^t e^{-\rho s}(dL(s) - \beta dG(s)) + e^{-\rho t}V(R^\pi(t))\Big].$$

Inserting Equation (28) in the equation above we obtain

$$\begin{aligned}
0 = \max_{\pi\in\Pi_p}\Bigg\{&\mathbb{E}\Big[\int_0^t e^{-\rho s}\mathcal{L}^a(V(R^\pi(s)))ds\Big]\\
&+\mathbb{E}\Big[\int_0^t e^{-\rho s}(1 - V'(R^\pi(s)))dL(s) + \int_0^t e^{-\rho s}(V'(R^\pi(s)) - \beta)dG(s)\Big]\\
&+\mathbb{E}\Big[\sum_{s\in\Gamma_1\cup\Gamma_2} e^{-\rho s}\big(\Delta V(R^\pi(s)) - V'(R^\pi(s))(\Delta R^\pi(s))\big)\Big]\Bigg\}.
\end{aligned} \tag{30}$$

If we write $\tilde{L}$ and $\tilde{G}$ for the continuous parts of L and G respectively, then the Equation (30) above may be rewritten as

$$\begin{aligned}
0 = \max_{\pi\in\Pi_p}\Bigg\{&\mathbb{E}\Big[\int_0^t e^{-\rho s}\mathcal{L}^a(V(R^\pi(s)))ds\Big]\\
&+\mathbb{E}\Big[\int_0^t e^{-\rho s}(1 - V'(R^\pi(s)))d\tilde{L}(s) + \int_0^t e^{-\rho s}(V'(R^\pi(s)) - \beta)d\tilde{G}(s)\Big]\\
&+\mathbb{E}\Big[\sum_{s\in\Gamma_1} e^{-\rho s}\big(\Delta V(R^\pi(s)) + \Delta L(s)\big)\Big] + \mathbb{E}\Big[\sum_{s\in\Gamma_2} e^{-\rho s}\big(\Delta V(R^\pi(s)) - \beta\Delta G(s)\big)\Big]\Bigg\}.
\end{aligned}$$

Notice that for all $s \in (0, t)$ it holds that

$$\Delta V(R^\pi(s)) + \Delta L(s) = \int_{R^\pi(s_-)-\Delta L(s)}^{R^\pi(s_-)}(1 - V'(x))dx \leq 0$$

and

$$\Delta V(R^\pi(s)) - \beta\Delta G(s) = \int_{R^\pi(s_-)}^{R^\pi(s_-)+\Delta G(s)}(V'(x) - \beta)dx \leq 0.$$

This implies all summands on the right hand side of Equation (30) are non positive, which concludes the proof.

Q.E.D.

Proof of Theorem 3.3. We have from Expression (10) that

$$V_c(x_0) = \frac{\beta}{\rho}\Big(\frac{\mu}{2}\Big)^{\gamma} c^{1-\gamma}$$

and

$$V_{x_1}(x_0) = \frac{1}{r_1 r_2}\Big(\frac{r_2^2}{r_2-r_1} e^{r_1 \frac{(1-\gamma)\sigma^2}{\mu}} e^{-r_1(\frac{\gamma}{\rho}c+x_1)} - \frac{r_1^2}{r_2-r_1} e^{r_2 \frac{(1-\gamma)\sigma^2}{\mu}} e^{-r_2(\frac{\gamma}{\rho}c+x_1)}\Big).$$

Let us define

$$A_1 := \frac{\beta}{\rho}\Big(\frac{\mu}{2}\Big)^{\gamma},\ A_2 := \frac{r_2}{r_1(r_2-r_1)} e^{r_1 \frac{(1-\gamma)\sigma^2}{\mu}} \text{ and } A_3 := \frac{r_1}{r_2(r_2-r_1)} e^{r_2 \frac{(1-\gamma)\sigma^2}{\mu}}.$$

Then the expression

$$F_1(c,x) := A_1 c^{1-\gamma} - A_2 e^{-r_1(\frac{\gamma}{\rho}c+x)} + A_3 e^{-r_2(\frac{\gamma}{\rho}c+x)} = 0$$

corresponds to the pasting condition $V_c(x_0) = V_{x_1}(x_0)$. Notice that $A_3 < A_2 < 0$, which implies that $F_1(0,0) = A_3 - A_2 < 0$. Next we have that

$$V_c'(x_0) = \beta\Big(\frac{\mu}{2}\Big)^{\gamma-1} c^{1-\gamma}$$

and

$$V_{x_1}'(x_0) = r_1 A_2 e^{-r_1(\frac{\gamma}{\rho}c+x_1)} - r_2 A_3 e^{-r_2(\frac{\gamma}{\rho}c+x_1)}.$$

If we define $A_1' := \beta\Big(\frac{\mu}{2}\Big)^{\gamma-1}$, then the condition we require is

$$F_2(c,x) := A_1' c^{1-\gamma} - r_1 A_2 e^{-r_1(\frac{\gamma}{\rho}c+x)} + r_2 A_3 e^{-r_2(\frac{\gamma}{\rho}c+x)} = 0.$$

Next we compute

$$F_2(c,x) - r_1 F_1(c,x) = (A_1' - r_1 A_1) c^{1-\gamma} + (r_2 - r_1) A_3 e^{-r_2(\frac{\gamma}{\rho}c+x)} = 0.$$

Solving for x in the expression above yields

$$x = -\frac{\gamma}{\rho}c - \frac{1}{r_2}\log\big(Bc^{1-\gamma}\big), \quad \text{where} \quad B := \frac{A_1' - r_1 A_1}{(r_1-r_2)A_3}. \tag{31}$$

Inserting Equation (31) into $F_1(c,x)$ yields

$$A_1 c^{1-\gamma} - A_2\big(Bc^{1-\gamma}\big)^{\frac{r_1}{r_2}} + A_3\big(Bc^{1-\gamma}\big) = 0;$$

therefore

$$c = \left(\frac{A_1 + BA_3}{A_2 B^{r_1/r_2}}\right)^{r_2/(1-\gamma)(r_1-r_2)}. \tag{32}$$

Substituting A_1, A_2, A_3 and B in Equation (31) we obtain

$$c(\beta) = \beta^{1/(\gamma-1)} A_5, \quad \text{where} \quad A_5 := \left(\frac{\tilde{A}_1 + \tilde{B}A_3}{A_2 \tilde{B}^{r_1/r_2}}\right)^{r_2/(1-\gamma)(r_1-r_2)}, \tag{33}$$

$\tilde{A}_i := \beta^{-1} A_i$ and $\tilde{B} := \beta^{-1} B$. Since $A_2 \tilde{B}^{r_1/r_2} < 0$, we must show that $\tilde{A}_1 + \tilde{B}A_3 < 0$ to have a well–defined $c(\beta)$. Observe that

$$\tilde{A}_1 + \tilde{B}A_3 = \tilde{A}_1 + \frac{\tilde{A}_1' - r_1\tilde{A}_1}{r_1 - r_2} = \frac{\tilde{A}_1' - r_2\tilde{A}_1}{r_1 - r_2}.$$

Since $r_1 - r_2 < 0$, $c(\beta)$ is well–defined whenever $\tilde{A}_1' - r_2\tilde{A}_1 > 0$. After simplification, the latter is equivalent to $2\rho > r_2\mu$, which in turn may be rewritten as

$$\frac{2\sigma^2\rho}{\mu} > -\mu + \sqrt{\mu^2 + 2\sigma^2\rho}. \tag{34}$$

Simple arithmetics are required to show that Expression (34) is equivalent to $\mu < \sqrt{\mu^2 + 2\sigma^2\rho}$, which holds for all $\mu, \sigma, \rho > 0$. Inserting $c(\beta)$ into Equation (31) we get

$$x = -\frac{\gamma}{\rho}c(\beta) - \frac{1}{r_2}\log\left(\left(-\frac{r_1}{r_2}\right)^{\frac{2r_2}{r_2-r_1}} e^{-r_2\frac{(1-\gamma)\sigma^2}{\mu}} \left[\frac{\tilde{A}_1' - \tilde{B}A_3}{\tilde{A}_1' - r_1\tilde{A}_1}\right]^{\frac{r_1}{(r_2-r_1)}}\right).$$

Thus,

$$\begin{aligned} x_1(\beta) &= -\frac{\gamma}{\rho}c(\beta) - \frac{1}{r_2}\log\left(e^{-r_2\frac{(1-\gamma)\sigma^2}{\mu}}\right) - \frac{1}{r_2}\log\left(\left(-\frac{r_1}{r_2}\right)^{\frac{2r_2}{r_2-r_1}} \left[\frac{\tilde{A}_1' - \tilde{B}A_3}{\tilde{A}_1' - r_1\tilde{A}_1}\right]^{\frac{r_1}{(r_2-r_1)}}\right) \\ &= \frac{\sigma^2(1-\gamma)}{\mu} - \frac{\gamma}{\rho}c(\beta) - \frac{1}{r_2}\log\left(\left(-\frac{r_1}{r_2}\right)^{\frac{2r_2}{r_2-r_1}} \left[\frac{\tilde{A}_1' - \tilde{B}A_3}{\tilde{A}_1' - r_1\tilde{A}_1}\right]^{\frac{r_1}{(r_2-r_1)}}\right) \\ &= x_0(\beta) + A_6, \end{aligned} \tag{35}$$

where

$$A_6 := -\frac{1}{r_2}\log\left(\left(-\frac{r_1}{r_2}\right)^{\frac{2r_2}{r_2-r_1}} \left[\frac{\tilde{A}_1' - \tilde{B}A_3}{\tilde{A}_1' - r_1\tilde{A}_1}\right]^{\frac{r_1}{(r_2-r_1)}}\right)$$

It should be remarked that A_6 coincides with $x_1 - x_0$ in the no–reinvestment case presented in Appendix **??**, thus $c(\beta)$ and $x_1(\beta)$ are such that $x_0(\beta) \le x_1(\beta)$.

Q.E.D.

Proof of Corollary 3.4. The expressions for $c(\beta)$ and $x_1(\beta)$ are given in Equations (33) and (35) respectively. The result for $V_{c(\beta)}(x_0(\beta))$ and $V_{c(\beta)}'(x_0(\beta))$ follows from the smooth pasting condition and the fact that

$$c(\beta)^{1-\gamma} = \beta^{-1}A_5^{1-\gamma} \quad \text{and} \quad \frac{\rho\sigma^2(1-\gamma)}{\gamma\mu} = \frac{\mu}{2}.$$

Q.E.D.

Proof of Proposition 3.5. Let $1 \le \beta_1 \le \beta_2$. Since $\gamma \in (0,1)$, then $\beta_2^1/(\gamma-1) < \beta_1^1/(\gamma-1)$. Hence, for $x \in [0, x_0(\beta_1)]$ we have

$$V_{c(\beta_1)}(x) = \frac{1}{\rho}A_5^{1-\gamma}\left(\frac{\rho}{\gamma}x + \beta_1^1/(\gamma-1)A_5\right) < \frac{1}{\rho}A_5^{1-\gamma}\left(\frac{\rho}{\gamma}x + \beta_2^1/(\gamma-1)A_5\right) = V_{c(\beta_2)}(x).$$

From Equations (10) and (35) we have that

$$V_{x_1(\beta)}(x) = \alpha_2 e^{-r_1x_0(\beta)}e^{r_1x} - \alpha_3 e^{-r_2x_0(\beta)}e^{r_2x},$$

where

$$\alpha_2 := \frac{r_2}{r_1(r_2-r_1)}e^{-r_1A_6} \quad \text{and} \quad \alpha_3 := \frac{r_1}{r_2(r_2-r_1)}e^{-r_2A_6}.$$

Notice that for $i = 1, 2$ $\alpha_i < 0$, and recall that $-r_2 < 0 < -r_1$. Since $x_0(\beta_1) < x_0(\beta_2)$ we have

$$\alpha_2 e^{-r_1 x_0(\beta_1)} > \alpha_2 e^{-r_1 x_0(\beta_2)} \quad \text{and} \quad -\alpha_3 e^{-r_2 x_0(\beta_1)} > -\alpha_3 e^{-r_2 x_0(\beta_2)}.$$

Thus, for $x \in [x_0(\beta_2), x_1(\beta_1)]$ we have $V_{x_1(\beta_1)}(x) > V_{x_1(\beta_2)}(x)$. From Corollary 3.4 we have that for $x \in [x_0(\beta_1), x_0(\beta_2)]$ it hold that $V_{c(\beta_2)}(x) < V_{x_1(\beta_1)}(x)$, and for $x \in [x_1(\beta_1), x_1(\beta_2)]$ we have $(x - x_1(\beta_1)) + \frac{r_2 + r_1}{r_1 r_2} > V_{x_1(\beta_2)}(x)$. We conclude that, for all $x \geq 0$, $V_{\beta_1}(x) > V_{\beta_2}(x)$.

Denote by x_1^* the $\lim_{\beta\to\infty} x_1(\beta)$ and by x_0^* the $\lim_{\beta\to\infty} x_0(\beta)$. These are, respectively, the dividend and the full–investment barriers for $U(\cdot)$. We know from Equation (35) that this is finite. For any $\epsilon > 0$, the family $\{V_\beta\}$ restricted to the domain $[\epsilon, x_1]$ is uniformly bounded. Moreover, all the V_β's are concave and their derivatives are uniformly bounded, hence $\{V_\beta\}$ is also a uniformly equicontinuous family. By the Arzelà–Ascoli theorem there exits a uniform limit, which we denote by V_∞. We have from Equation (33) that

$$\lim_{\beta\to\infty} V_\beta(0) = \lim_{\beta\to\infty} \frac{\beta}{\rho} \beta^{1/(\gamma-1)} A_5 = 0.$$

Furthermore, the net $\{V_\beta(0)\}$ decreases monotonically with β, which allows us to conclude that, even setting $\epsilon = 0$, the family $\{V_\beta\}$ has V_∞ as a limit. Since $x_1(\beta) \to x_1^*$, we have from Equation (6) that $V_{x_1(\beta)} \to U$ as the domains of the $V_{x_1(\beta)}$'s converges (in terms of inclusions) to $[x_0^*, x_1^*]$. On $[0, x_0^*]$, V_∞ is a radical function of exponent γ. $V_\infty(0) = 0$ and $V_\infty(x_0^*) = A_5(\mu, \sigma, \rho)^{1-\gamma} \frac{1}{\rho} \left(\frac{\mu}{2}\right)^\gamma$. This coincides with the solution to the boundary–value problem that generates $U(x)$, and we may conclude that $\{V_\beta\}$ converges uniformly to U as β grows to infinity. The proof that $V_\beta(x)$ converges uniformly to $x + \rho^{-1} A_5$ as β tends to one is analogous.

Q.E.D.

Proof of Theorem 3.6. Consider an arbitrary strategy $\pi = (a, L, G) \in \Pi_p$, and an initial level of cash reserves $x \geq 0$. Recall that the corresponding cash–reserves process evolves according to the SDE

$$dR^\pi(t) = a(t)(\mu dt + \sigma dW(t)) - dL(t) + dG(t),\ R^\pi(0) = x.$$

Proceeding as in the proof of Proposition 3.1 we use (the generalized) Itô's formula applied to $f(t, x) = e^{-\rho t} V_\beta(x)$ and obtain, after simplifications (recall that $\mathcal{L}^a V_\beta(R^\pi(t)) \leq 0$)

$$\begin{aligned} e^{-\rho t}\mathbb{E}\left[V_\beta(R^\pi(t))\right] \leq V_\beta(x) - \mathbb{E}\left[\int_0^t e^{-\rho s} V_\beta'(R^\pi(s))\big(d\tilde{L}(s) - d\tilde{G}(s)\big)\right] \\ + \mathbb{E}\left[\sum_{s\in\Gamma_1\cup\Gamma_2} e^{-\rho s}\big(V_\beta(R^\pi(s)) - V_\beta(R^\pi(s_-))\big)\right], \end{aligned} \tag{36}$$

where

$$\tilde{L}(s) := L(s) - L(s_-) \quad \text{and} \quad \tilde{G}(s) := G(s) - G(s_-)$$

are the continuous parts of L and G respectively. If $s \in \Gamma_1 \cap \Gamma_2$, then redefining $L(s) := (L(s) - G(s))_+$ and $G(s) := (G(s) - L(s))_+$ results in the same strategy with a jump only for L or G. We may therefore assume without loss of generality that Γ_1 and Γ_2 are disjoint. Let $s \in \Gamma_1$, then by the Mean Value Theorem and the fact that $1 \leq V_\beta'(R^\pi(s)) \leq \beta$, there exists $r^* \in (R^\pi(s_-), R^\pi(s))$ such that

$$V_\beta(R^\pi(s)) - V_\beta(R^\pi(s_-)) = V_\beta'(r^*)\big(R^\pi(s) - R^\pi(s_-)\big) \geq -(L(s) - L(s_-)).$$

Analogously, if $s \in \Gamma_2$, we have

$$V_\beta(R^\pi(s)) - V_\beta(R^\pi(s_-)) = V_\beta'(r^*)\big(R^\pi(s) - R^\pi(s_-)\big) \geq \beta(G(s) - G(s_-)).$$

Inserting the above expressions into Expression (36) we get

$$e^{-\rho t}\mathbb{E}\left[V_\beta(R^\pi(t))\right] \leq V_\beta(x) - \mathbb{E}\left[\int_0^t e^{-\rho s}\big(dL(s) - \beta dG(s)\big)\right].$$

By continuity $V_\beta(x)$ is bounded for $x \in [0, x_1(\beta)]$ and it grows linearly as x tends to infinity, therefore

$$\lim_{t\to\infty} e^{-\rho t}\mathbb{E}\left[V_\beta(R^\pi(t))\right] = 0.$$

This implies that

$$V_\beta(x) \geq \mathbb{E}\left[\int_0^\infty e^{-\rho s}\big(dL(s) - \beta dG(s)\big)\right]. \tag{37}$$

Next consider the strategy $\pi^* = (a^*, L^*, G^*)$. Since L^* and G^* are the local times of R^* at levels $x_1(\beta)$ and 0 respectively, we may assume that $x \in [0, x_1(\beta)]$. Furthermore, L^* and G^* are continuous processes, and on $[0, x_1(\beta)]$ it holds that $\mathcal{L}^a(V_\beta(R^\pi(s))) = 0$. Hence, for the strategy π^* Itô's formula yields

$$e^{-\rho t}V_\beta(R^\pi(t)) = V_\beta(x) + \int_0^t e^{-\rho s}V_\beta'(R^\pi(s))dW(s) - \int_0^t e^{-\rho s}V_\beta'(R^\pi(s))\big(dL^*(s) - dG^*(s)\big). \tag{38}$$

The measure $dL^*(s)$ is supported on $\{R^\pi(s) = x_1(\beta)\}$ and $V_\beta'(x_1(\beta)) = 1$. The measure $dG^*(s)$ is supported on $\{R^\pi(s) = 0\}$ and $V_\beta'(0) = \beta$, therefore, taking expectations, Equation (38) may be rewritten as

$$e^{-\rho t}\mathbb{E}\left[V_\beta(R^\pi(t))\right] = V_\beta(x) - \mathbb{E}\left[\int_0^t e^{-\rho s}\big(dL(s) - \beta dG(s)\big)\right].$$

Letting $t \to \infty$ we have

$$V_\beta(x) = \mathbb{E}\left[\int_0^\infty e^{-\rho s}\big(dL(s) - \beta dG(s)\big)\right]. \tag{39}$$

Form Equation (37) we have that for any $\pi \in \Pi_p$, $V_\beta(x) \geq V^\pi(x)$. Since $\pi^* \in \Pi_p$ Equation (39) yields

$$V_\beta(x) = \sup_{\pi\in\Pi_p} V^\pi(x),$$

which concludes the proof.

Q.E.D.

Proof of Proposition 3.7. We proceed along the lines of the proof of Theorem 3.3. Reflecting at level $\overline{x} > 0$ implies that $\overline{V}'_\beta(\overline{x}) = \beta$. The latter yields, for $x \in [\overline{x}, \overline{x}_0]$,

$$\overline{V}_\beta(x) = \frac{\beta}{\rho}\left(\frac{\rho}{\gamma}\overline{x} + \overline{c}(\beta)\right)^{1-\gamma}\left(x\frac{\rho}{\gamma} + \overline{c}(\beta)\right)^{\gamma}.$$

In this case we have

$$\overline{c}(\beta) = \beta^{1/(\gamma-1)}A_5 - \frac{\rho}{\gamma}\overline{x}.$$

In other words, switching from the reflection–at–zero regime to the reflection–at–$\overline{x}$ one reduces $c(\beta)$ by $\frac{\rho}{\gamma}\overline{x}$, which in turn impacts the level of risk exposure in the following way:

$$\overline{a}(x) = a(x) - \frac{2\rho}{\gamma\mu}\overline{x}.$$

The smooth–pasting conditions then result in $\overline{x}_0(\beta) = x_0(\beta) + \overline{x}$ and $\overline{x}_1(\beta) = x_1(\beta) + \overline{x}$. Notice that

$$\begin{aligned}
\overline{V}_\beta(\overline{x}) = V_\beta(0) &= \frac{A_5}{\rho}\beta^{\gamma/(\gamma-1)};\\
\overline{V}_\beta(\overline{x}_0(\beta)) = V_\beta(x_0(\beta)) &= \frac{A_5^{1-\gamma}}{\rho}\left(\frac{\mu}{2}\right)^\gamma \beta^{\gamma/(\gamma-1)};\\
\overline{V}_\beta(\overline{x}_1(\beta)) = V_\beta(x_1(\beta)) &= \frac{\mu}{\sigma^2\rho}.
\end{aligned}$$

In fact we have that, for $x \in [\overline{x}, \infty)$, $\overline{V}_\beta(x) = V_\beta(x - \overline{x})$. This clearly shows that $\overline{V}_\beta$ is strictly dominated by V_β whenever $x \geq \overline{x}$.

Q.E.D.

Proof of Proposition 4.2. In analogous fashion as in the proof of Proposition 3.1, we have that

$$0 \geq \max_{a\in[0,1]}\left\{\frac{1}{2}\sigma^2 a^2 V'' + \mu a V' - \rho V\right\} \quad \text{and} \quad 0 \geq 1 - V'(x).$$

Given a reserves level x, the value $H(x, V) = \sup_{u\geq 0}\left\{V(x+u) - u - \kappa\right\}$ corresponds to executing the best immediate action (in terms of cash injection) and behaving optimally after that. Since there are states where it might not be optimal to exercise the option of refinancing, we have $V(x) \geq H(x, V)$, or $0 \geq H(x, V) - V(x)$. The fact that one of these three inequalities binds follows from argument analogous to those found in the proof of Proposition 3.1.

Proof of Proposition 4.6. Inserting Expression (22) into $V_c(x)$ we have that

$$V_c(x_0) = c\Big(\frac{\mu}{2}\Big)^\gamma \quad \text{and} \quad V_c'(x_0) = c\rho\Big(\frac{\mu}{2}\Big)^{\gamma-1}.$$

Inserting Expression (22) into $V_{x_1}(x)$ yields

$$V_{x_1}(x_0) = B_2 e^{-r_1(\frac{\gamma}{\rho}\mathbf{s}(c,x_1)+x_1)} - B_3 e^{-r_2(\frac{\gamma}{\rho}\mathbf{s}(c,x_1)+x_1)},$$

where

$$B_2 := \frac{r_2}{r_1(r_2 - r_1)} e^{r_1\frac{\gamma\mu}{2\rho}} \text{ and } B_3 := \frac{r_1}{r_2(r_2 - r_1)} e^{r_2\frac{\gamma\mu}{2\rho}}.$$

Furthermore

$$V_{x_1}'(x_0) = r_1 B_2 e^{-r_1(\frac{\gamma}{\rho}\mathbf{s}(c,x_1)+x_1)} - r_2 B_3 e^{-r_2(\frac{\gamma}{\rho}\mathbf{s}(c,x_1)+x_1)}.$$

The smooth–pasting conditions result in system of equations on c and x. Proceeding as in the proof of Theorem 3.3, we obtain that x is implicitly determined by the equation

$$x = -\frac{1}{r_2}\log\left(c\Big(\frac{\mu}{2}\Big)^{\gamma-1}\big(\rho - r_1\mu/2\big)/(r_1 - r_2)B_3\right) - \frac{\gamma}{\rho}\mathbf{s}(c, x). \tag{40}$$

Inserting Expression (40) into the smoothness condition $V_c'(x_0) - V_x'(x_0) = 0$ yields

$$c\left[\Big(\frac{\mu}{2}\Big)^\gamma - B_3\frac{r_2}{r_1}\Big(\frac{\mu}{2}\Big)^{\gamma-1}(\rho - r_1\mu/2)e^{-r_2\frac{\gamma\mu^2}{\sigma^2\rho}}\right] = B_2\left[-\frac{r_2}{r_1}c\Big(\frac{\mu}{2}\Big)^{\gamma-1}(\rho - r_1\mu/2)e^{-r_2\frac{\gamma\mu^2}{\sigma^2\rho}}\right]^{\frac{r_1}{r_2}}$$

Factorizing c we obtain

$$c = 2\Big(-\frac{r_2}{r_1}\Big)^{\frac{r_2+r_1}{r_2-r_1}}\Big(\frac{\mu}{2}\Big)^{1-\gamma}\left[\frac{(2\rho-r_1\mu)^{r_1}}{(2\rho-r_2\mu)^{r_2}}\right]^{\frac{1}{r_2-r_1}}.$$

Define $A := \sqrt{\mu^2+2\sigma^2\rho}$ and observe that $2\rho - r_i\mu = (-1)^i r_i A$, which gives

$$c = \frac{2}{A}\Big(-\frac{r_2}{r_1}\Big)^{\frac{r_2+r_1}{r_2-r_1}}\Big(\frac{\mu}{2}\Big)^{1-\gamma}\left[\frac{-r_1^{r_1}}{r_2^{r_2}}\right]^{\frac{1}{r_2-r_1}}.$$

Next we write

$$\frac{-r_1^{r_1}}{r_2^{r_2}} = \frac{(-r_1)^{-\mu/\sigma^2}(-r_1)^{-A/\sigma^2}}{(r_2)^{-\mu/\sigma^2}(r_2)^{A/\sigma^2}} = \Big(-\frac{r_1}{r_2}\Big)^{-\mu/\sigma^2}(-r_1r_2)^{-A/\sigma^2},$$

which together with the fact that $r_2 - r_1 = 2A/\sigma^2$ implies

$$c = \frac{2}{A}\Big(-\frac{r_2}{r_1}\Big)^{\frac{r_2+r_1}{r_2-r_1}}\Big(\frac{\mu}{2}\Big)^{1-\gamma}\Big(-\frac{r_2}{r_1}\Big)^{\mu/2A}(-r_1r_2)^{1/2} = \frac{2}{A\sqrt{-r_1r_2}}\Big(-\frac{r_2}{r_1}\Big)^{-\mu/2A}\Big(\frac{\mu}{2}\Big)^{1-\gamma}. \quad (41)$$

Inserting Equation (41) into Equation (40) yields after simplification the required expression for $\frac{\gamma}{\rho}\mathbf{s}(c,x) + x$.

Q.E.D.

Proof of Theorem 4.7. Para paraphrasing the first part of the proof of Theorem 3.6, one can show without much difficulty that for an arbitrary strategy $\pi = (a, L, \hat{x}) \in \Pi_f$, an initial level of cash reserves $x \geq 0$ and a fixed–cost κ it holds that

$$V_\kappa(x) \geq \mathbb{E}\left[\int_0^\infty e^{-\rho s}\big(dL(s) - (\hat{x}+\kappa)dJ(s)\big)\right], \quad (42)$$

where the process $\{J\}$ is as defined in Expression (17). Next consider the strategy $\pi^* = (a^*_\kappa, L^*, x^*_1(\kappa))$. By construction we may assume that $x \in [0, x^*_1(\kappa)]$, a region in which it holds that $\mathcal{L}^a(V_\kappa(R^\pi(s))) = 0$. Hence, for the strategy π^* Itô's formula yields

$$\begin{aligned} e^{-\rho t}V_\kappa(R^\pi(t)) = V_\kappa(x) &+ \int_0^t e^{-\rho s}V'_\kappa(R^\pi(s))dW(s) - \int_0^t e^{-\rho s}V'_\kappa(R^\pi(s))\big(d\tilde{L}(s) - x^*_1(\kappa)d\tilde{J}(s)\big) \\ &+ \sum_{s\in\Gamma} e^{-\rho s}\big(V_\kappa(R^\pi(s)) - V_\kappa(R^\pi(s_-))\big), \end{aligned}$$

where Γ is the set of discontinuities of $\{R^\pi\}$ and $\tilde{L}(s)$ and $\tilde{J}(s)$ are the continuous parts of the processes $\{L\}$ and $\{J\}$. Since the former is itself continuous and the latter is a pure jump process we may write

$$\begin{aligned} e^{-\rho t}V_\kappa(R^\pi(t)) = V_\kappa(x) &+ \int_0^t e^{-\rho s}V'_\kappa(R^\pi(s))dW(s) - \int_0^t e^{-\rho s}V'_\kappa(R^\pi(s))dL(s) \\ &+ \sum_{s\in\Gamma} e^{-\rho s}\big(V_\kappa(R^\pi(s)) - V_\kappa(R^\pi(s_-))\big). \end{aligned} \quad (43)$$

The set Γ corresponds, in fact, to the crossings of level zero by $\{R^\pi\}$. Therefore for $s \in \Gamma$ we have (from the non–local boundary condition at zero)

$$V_\kappa(R^\pi(s)) - V_\kappa(R^\pi(s_-)) = V_\kappa(x^*_1(\kappa)) - \big(V_\kappa(x^*_1(\kappa)) - x^*_1(\kappa) - \kappa\big) = x^*_1(\kappa) + \kappa.$$

The measure $dL^*(s)$ is supported on $\{R^\pi(s) = x_1(\beta)\}$ and $V'_\beta(x_1(\beta)) = 1$. therefore, taking expectations, Equation (43) becomes

$$e^{-\rho t}\mathbb{E}\left[V_\kappa(R^\pi(t))\right] = V_\kappa(x) - \mathbb{E}\left[\int_0^t e^{-\rho s}\big(dL^*(s) - (x_1^*(\kappa)+\kappa)dJ(s)\big)\right].$$

Notice that the expectation of the Itô integral is zero since for all $s \geq 0$ it holds that $V'_\kappa(R^\pi(s)) \in \big[1, \rho c^*(\mathbf{s}(c^*, x_1^*(\kappa)))^{-\mu/2\sigma^2\rho}\big]$. Letting $t \to \infty$ we have

$$V_\kappa(x) = \mathbb{E}\left[\int_0^\infty e^{-\rho s}\big(dL^*(s) - (x_1^*(\kappa)+\kappa)dJ(s)\big)\right],$$

which concludes the proof.

Q.E.D.

Appendix B. The Structure of the Value Function in [7]

We include a quick description of the structure of the value function in [7] for completeness. This corresponds to the solution of the HJB equation

$$\max\left\{\max_{\alpha\in[0,1]}\left\{\frac{1}{2}\sigma^2\alpha^2U'' + \mu\alpha U' - \rho V\right\}, 1 - U'(x)\right\} = 0.$$

Proceeding as in Section 3 we have that On $(0, x_0)$,

$$U(x) = c_1\left(x\frac{\rho}{\gamma} + c_2\right)^\gamma$$

In this case, however, $U(0) = 0$, which implies $c_2 = 0$. As a consequence

$$\alpha(x) = -\frac{\mu}{\sigma^2}\frac{U'(x)}{U''(x)} = -\frac{\mu}{\sigma^2}\Big(c\rho\big(x\frac{\rho}{\gamma}\big)^{\gamma-1}\Big)/\Big(\frac{c\rho^2(\gamma-1)}{\gamma}\big(x\frac{\rho}{\gamma}\big)^{\gamma-2}\Big) = -\frac{\mu}{\sigma^2}\frac{x}{\gamma-1}.$$

Equating $a(x) = 1$ then yields

$$x_0 = \frac{\sigma^2(1-\gamma)}{\mu}.$$

As previously mentioned, in this case x_0 is not a free boundary. On (x_0, x_1) and $[x_1, \infty)$ things remain unchanged. Thus, the structure of the value function is

$$U(x) = \begin{cases} c^*\left(x\frac{\rho}{\gamma}\right)^\gamma, & x \in [0, x_0); \\ \frac{1}{r_1r_2}\Big(\frac{r_2^2}{r_2-r_1}e^{r_1(x-x_1^*)} - \frac{r_1^2}{r_2-r_1}e^{r_2(x-x_1^*)}\Big), & x \in [x_0, x_1^*); \\ (x - x_1^*) + \frac{r_2+r_1}{r_1r_2}, & x \in [x_1^*, \infty). \end{cases}$$

Here r_i is as before, and the pair (c^*, x_1^*) is chosen as to satisfy the smooth–pasting conditions, as to have $V \in \mathcal{C}^2$.

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